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Hilbert space of space-time SCFT in AdS_3 Superstring and T^{4kp}/S_{kp} SCFT

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Abstract: We explore the superstring theory on $AdS_3 \times S^3 \times T^4$ in the framework given in [1]. We argue on the Hilbert space of *space-time CFT*, and especially construct a suitable vacuum of this CFT from the physical degrees of freedom of the superstring theory in bulk. We first construct it explicitly in the case of $p = 1$, and then present a proposal for the general cases of $p > 1$. After giving some completion of the GKS's constructions of the higher mode operators (in particular, of those including spin fields), we also make some comparison between the space-time CFT and T^{4kp}/S_{kp} SCFT, namely, with respect to the physical spectrum of chiral primaries and some algebraic structures of bosonic and fermionic oscillators in both theories. We also observe how our proposal about the Hilbert space of space-time CFT leads to a satisfactory correspondence between the spectrum of chiral primaries of both theories in the cases $p > 1$.

Keywords: String Duality, Conformal Field Models in String Theory.

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Contents

1. Introduction

In the recent studies of the proposed duality between the supergravity theory (SUGRA) on $AdS_{d+1} \times K$ and the d-dimensional conformal field theory (CFT) defined on the boundary of AdS_{d+1} [2, 3, 4], the works on AdS_3 have got a special status. Both of the AdS_3-gravity and 2-dimensional CFT have infinite-dimensional local symmetries [5], which bring us a high ability of calculation. Thanks to this fact, several fruitful results related with the black-hole physics are given [6, 7, 8, 9, 10, 11, 12]. Among other things, it is remarkable that we can expect rich possibilities to establish the AdS/CFT-duality in *stringy* level. As is well-known, a classical solution of SUGRA corresponds to a 2-dimensional CFT on the world-sheet of first quantized superstring. Therefore the AdS_3/CFT_2-duality in stringy level naturally leads to the correspondence between the two 2-dimensional CFTs - the " world-sheet CFT" and the *space-time CFT*,[1] the latter of which is defiend on the boundary of AdS space.

[1]In many litarature, the terminology *boundary CFT* is often used in this sense. Throughout this paper, we shall use the terminology *space-time CFT* in the meaning of the one proposed in [1] as a candidate for the boundary CFT.

Among other things Giveon, Kutasov and Seiberg (GKS) have obtained a remarkable result [1]: The typical operators (the generators of superconformal algebra) in the space-time CFT have been given directly from the physical degrees of freedom of the world-sheet CFT. Recently, there are some further developments [13, 14, 15, 16] along this line.

However, there still exists an important question which is not yet answered: What is the Hilbert space of space-time CFT? Especially, what is the suitable vacuum on which the GKS's operators in space-time CFT should act? Needless to say, answering this question is significant for calculating correlators in the space-time CFT (and of course, according to the standard programs in recent AdS-business, comparing them with those calculated by SUGRA or string theory in bulk...). Solving this problem is one of the main purposes of this paper. We will propose an explicit realization of the *space-time vacuum*.

Another important subject which may be interested by many theoretists studying AdS/CFT-duality is to confirm the equivalence (or some relationship) between the space-time CFT of GKS and the SCFT on the symmetric orbifold T^{4kp}/S_{kp}, which is derived by some brane dynamics. We carefully investigate the spectrum of chiral primary fields in space-time CFT, and discuss the equivalence with that of T^{4kp}/S_{kp}-SCFT.

This paper is organized as follows: We shall begin in section 2 by fixing our convetion and giving a short review of the formulation of GKS for convenience of readers. We will also present some non-trivial completion of the calculation of GKS (all of the explicit forms of the higher modes of supercurrents).

In section 3, we discuss the desired vacuum of space-time CFT. We first construct it explicitly for the case of $p = 1$, (p is equal to the NS1 charge in the set up in [1].) and then we propose a candidate of the Hilbert space with a suitable vacuum for the general case of $p > 1$.

In section 4, in order to study the relation between the space-time CFT and T^{4kp}/S_{kp}-SCFT, we start by working out all the bosonic and fermionic oscillators along T^4 in the space-time CFT that acts properly on the vacuum defined in section 3. Making use of this knowledge, we construct all of the mode operators of typical chiral primary fields, and discuss about the equivalence with the physical spectrum of T^{4kp}/S_{kp}-SCFT. Especially, we will show how our proposal of the Hilbert space can resolve the superficial contradictions about the chiral primary spectrum for the general case $p > 1$. We also comment on the relation between the SCA given by GKS and another SCA which is constructed as the quadratic forms of the bosonic and fermionic oscillators along T^4.

Section 5 is devoted to give some comments on several open problems.

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2. Superstring on $AdS_3 \times S^3 \times T^4$

Type IIB superstring theory on the spacetime $AdS_3 \times S^3 \times T^4$ with NS1 and NS5-brane charges was investigated in [1]. One of the main results there was that one can construct an $N = 4$ SCA, which should act on the boundary of AdS_3, out of certain physical operators in the world-sheet theory. This SCA is considered to be identified with the one discovered by Brown and Henneaux [5], which emerges when one considers the (super)gravity theory on a three-dimensional space-time with a negative cosmological constant. This aspect is also realized elegantly from the viewpoint of the 3-dimensional Chern-Simons gravity [7, 9].

Let us give a brief review of the work [1] here. There is a well-known solution in type IIB supergravity which is identified with the bound state of some NS1 and NS5-branes[17, 18, 19]. If we approach the near-horizon region of this solution, the geometry reduces to that of $AdS_3 \times S^3 \times \mathcal{M}^4$, where $\mathcal{M}^4$ is some four-dimensional spatial manifold. The AdS_3 part has the following metric:

$$ds^2 = l^2 \left(d\varphi^2 + e^{2\varphi} d\gamma d\bar{\gamma} \right) , \qquad l^2 = l_s^2 k' , \tag{2.1}$$

where l_s is the string length and k' is an integer which is roughly regarded as the number of NS5-branes. Taking the presence of NS-NS B-field into account, one can write down the action of the string on AdS_3-space. After evaluating some quantum corrections it becomes as

$$\mathcal{L} = \partial\varphi\bar{\partial}\varphi - \frac{2}{\alpha_+}\hat{R}^{(2)}\varphi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} - \beta\bar{\beta}\exp\left(-\frac{2\varphi}{\alpha_+}\right) , \tag{2.2}$$

where $\alpha_+ = \sqrt{2k' - 4}$. The worldsheet theory has two copies of $SL(2, R)$ current algebra of level k' (the left mover and right mover). These are represented in the standard form of Wakimoto construction [20]:

$$\begin{aligned} j^- &= \beta \, , \\ j^3 &= \beta\gamma + \frac{\alpha_+}{2}\partial\varphi \, , \\ j^+ &= \beta\gamma^2 + \alpha_+\gamma\partial\varphi + k'\partial\gamma \, , \end{aligned} \tag{2.3}$$

together with the OPE's of the fields

$$\varphi(z)\varphi(w) \sim -\ln(z - w) \, , \qquad \beta(z)\gamma(w) \sim \frac{1}{z - w} \, . \tag{2.4}$$

Of course, the right mover can be written in the same form.

One can apply, at least formally, the same argument to the S^3 part, which yields the following expression for $SU(2)$ current algebra of level k'':

$$\begin{aligned} k^- &= \tilde{\beta}\,, \\ k^3 &= \tilde{\beta}\tilde{\gamma} + \frac{i\alpha_+}{2}\partial\tilde{\varphi}\,, \\ k^+ &= -\tilde{\beta}\tilde{\gamma}^2 - i\alpha_+\tilde{\gamma}\partial\tilde{\varphi} + k''\partial\tilde{\gamma}\,, \end{aligned} \tag{2.5}$$

where $\alpha_+ = \sqrt{2k''+4}$, together with the following OPE's

$$\tilde{\varphi}(z)\tilde{\varphi}(w) \sim -\ln(z-w)\,, \qquad \tilde{\beta}(z)\tilde{\gamma}(w) \sim \frac{1}{z-w}\,. \tag{2.6}$$

The constant α_+ in the above must be the same as the one in the AdS_3 part. It may be convenient to note the relation;

$$\frac{\alpha_+^2}{2} = k' - 2 = k'' + 2 \equiv k\,. \tag{2.7}$$

To supersymmetrize the worldsheet theory one only has to add some free fermions ψ^A, χ^a which have the following OPE's[2]

$$\psi^A(z)\psi^B(w) \sim \frac{\eta^{AB}}{z-w}\,, \qquad \eta^{AB} = \mathrm{diag}(++-) \tag{2.8}$$

$$\chi^a(z)\chi^b(w) \sim \frac{\delta^{ab}}{z-w} \tag{2.9}$$

to the bosonic theory. Using these fermions one can construct $SL(2,R)$ and $SU(2)$ currents of levels -2 and 2, respectively,

$$j_\psi^\pm = \pm\psi^\pm\psi^3\,, \qquad j_\psi^3 = \frac{1}{2}\psi^+\psi^-\,, \qquad \psi^\pm = \psi^1 \pm i\psi^2\,, \tag{2.10}$$

$$k_\chi^\pm = \mp\chi^\pm\chi^3\,, \qquad k_\chi^3 = \frac{1}{2}\chi^+\chi^-\,, \qquad \chi^\pm = \chi^1 \pm i\chi^2\,, \tag{2.11}$$

The total currents are defined as

$$J^A = j^A + j_\psi^A\,, \qquad K^a = k^a + k_\chi^a\,, \tag{2.12}$$

both of which are of level $k = k' - 2 = k'' + 2$. Furthermore we introduce four bosons Y^i and four fermions λ^i for T^4 degrees of freedom. The worldsheet theory has the following energy-momentum tensor $T(z)$ and the supercurrent $G(z)$:

$$T(z) = \frac{1}{k}\left(j^A j_A + k^a k^a\right) - \frac{1}{2}\left(\psi^A\partial\psi_A + \chi^a\partial\chi^a + \partial Y^i\partial Y^i + \lambda^i\partial\lambda^i\right), \tag{2.13}$$

$$G(z) = \sqrt{\frac{2}{k}}\left(\psi^A j_A - \frac{i}{6}\epsilon^{ABC}\psi_A\psi_B\psi_C + \chi^a k^a - \frac{i}{6}\epsilon^{abc}\chi^a\chi^b\chi^c\right) + i\lambda^i\partial Y^i\,, \tag{2.14}$$

[2]Although our convention about the signature of metric η^{AB} is the same as the one given in [1], it is more natural in a physical reason to take the inverse sign for it. In this case, we must suppose that the bosonic $SL(2,\mathbf{R})$-current should have the level $-k'$, and the fermionic $SL(2,\mathbf{R})$-current should have the level $+2$. We should thank I. Bars for his comment on this fact.

In [1] the *space-time* $N=4$ SCA was constructed out of vertex operators in the world-sheet theory. The Virasoro operators $\mathcal{L}_n$ and the mode operators of $SU(2)$ current $\mathcal{K}^a_n$ have the following form:

$$\mathcal{L}_n = -\sqrt{\frac{k}{2}}\oint dz e^{-\phi}\left[(1-n^2)\psi^3\gamma^n + \frac{n(n-1)}{2}\psi^-\gamma^{n+1} + \frac{n(n+1)}{2}\psi^+\gamma^{n-1}\right], \quad (2.15)$$

$$\mathcal{K}^a_n = \sqrt{\frac{k}{2}}\oint dz e^{-\phi}\chi^a\gamma^n\,, \quad (2.16)$$

where ϕ is the bosonized super-reparametrization ghost. In the above expressions we are working in the (-1)-picture and we can obtain the expression in the 0-picture by the standard picture-changing operation [21]. The fermionic generators are constructed in terms of the spin fields of the worldsheet theory. In order to obtain the explicit forms of these generators we first give the bosonization formula of fermions:

$$\begin{aligned}
\sqrt{2}\,e^{\pm iH_1} &\equiv \psi^{\pm 1} \equiv \psi^1 \pm i\psi^2\,,\\
\sqrt{2}\,e^{\pm iH_2} &\equiv \psi^{\pm 2} \equiv \chi^1 \pm i\chi^2\,,\\
\sqrt{2}\,e^{\pm iH_3} &\equiv \psi^{\pm 3} \equiv \chi^3 \pm \psi^3\,,\\
\sqrt{2}\,e^{\pm iH_4} &\equiv \psi^{\pm 4} \equiv \lambda^1 \pm i\lambda^2\,,\\
\sqrt{2}\,e^{\pm iH_5} &\equiv \psi^{\pm 5} \equiv \lambda^3 \pm i\lambda^4\,, \quad (2.17)
\end{aligned}$$

up to some cocycle factors that ensure all the fermions to be mutually anti-commuting. Then we define the spin fields $S^{\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5}(z)=S^A(z)$, where ϵ_I takes values ± 1, as

$$S^A(z) \equiv \exp\left(\frac{i}{2}\sum_I \epsilon_I H_I(z)\right)\,, \quad (2.18)$$

up to cocycle factors. The calculations including cocycle factors are rather complicated, and are not discussed here. One only has to remember that it is defined consistently with the OPE's of fermions and spin fields. We summarize the relevant formulas in the appendix.

The fermionic generators in SCA can be written in terms of spin fields. Although only the explicit forms of the *zero-mode* part of these (corresponding to the global space-time SUSY) are given in [1], we can now write down the general higher modes of supercurrents:

$$\begin{aligned}
\mathcal{G}^{+\epsilon}_r &\equiv \oint dz e^{-\frac{\phi}{2}}\left[\left(r+\frac{1}{2}\right)\gamma^{r-\frac{1}{2}}S^{++-\epsilon\epsilon}(z) - \left(r-\frac{1}{2}\right)\gamma^{r+\frac{1}{2}}S^{-++\epsilon\epsilon}(z)\right],\\
\mathcal{G}^{-\epsilon}_r &\equiv \oint dz e^{-\frac{\phi}{2}}\left[\left(r+\frac{1}{2}\right)\gamma^{r-\frac{1}{2}}S^{+-+\epsilon\epsilon}(z) - \left(r-\frac{1}{2}\right)\gamma^{r+\frac{1}{2}}S^{---\epsilon\epsilon}(z)\right]. \quad (2.19)
\end{aligned}$$

In particular, for $r=\pm 1/2$ we have

$$\mathcal{G}^{+\epsilon}_{\frac{1}{2}} \equiv \oint dz e^{-\frac{\phi}{2}}S^{++-\epsilon\epsilon}(z)\,, \qquad \mathcal{G}^{+\epsilon}_{-\frac{1}{2}} \equiv \oint dz e^{-\frac{\phi}{2}}S^{-++\epsilon\epsilon}(z)\,,$$

$$\mathcal{G}^{-\epsilon}_{\frac{1}{2}} \equiv \oint dz e^{-\frac{\phi}{2}} S^{+-+\epsilon\epsilon}(z) \ , \qquad \mathcal{G}^{-\epsilon}_{-\frac{1}{2}} \equiv \oint dz e^{-\frac{\phi}{2}} S^{---\epsilon\epsilon}(z) \ , \tag{2.20}$$

which, of course, are the same as the original result given in [1].

Recently, the similar expressions for the supercurrents based on some affine Lie superalgebra were given [13]. It may be interesting to study the relation between them and our results here (2.19).

Note that we are restricting ourselves to the spin fields of definite chirality, namely, $\Pi\epsilon_I = -1$, due to the requirement of mutual locality of supercharges. There is a further restriction coming from the BRST invariance: for example, of all the operators which have the form

$$\oint dz e^{-\frac{\phi}{2}} S^A \ ,$$

only those with $\epsilon_1\epsilon_2\epsilon_3 = -1$ are BRST invariant.[3]

After some straightforward but lengthy calculations, we can directly show that the operators $\{\mathcal{L}_n,\, \mathcal{K}^a_n,\, \mathcal{G}^{\alpha A}\}$ actually generate the correct $N = 4$ SCA with central charge $c = 6pk$, where we impose the constraints $\oint dz\gamma^{-1}\partial\gamma = p$. As was discussed in [1], this condition is the essential part to reproduce the correct central term in SCA.

There is one subtlety which might lead one to a misunderstanding. This condition might sound peculiar from the viewpoints of usual string theory. In fact, as was claimed in the recent paper [16], it is reasonable to think that $\oint dz\gamma^{-1}\partial\gamma = 0$ is realized on the correct string vacuum. In [16] it was further argued that the correct central term in the Virasoro algebra of the boundary CFT should arise from the contribution of non-connected world-sheets, not from $\oint dz\gamma^{-1}\partial\gamma = p$. At first sight this argument might seem to contradict with the treatment of GKS. But this is not the case. One should consider that the equality

$$\oint dz\gamma^{-1}\partial\gamma \ |0\rangle = 0 \tag{2.21}$$

holds on the correct vacuum of string theory on AdS_3-background, and at the same time, should suppose that the condition

$$\oint dz\gamma^{-1}\partial\gamma \,|\mathrm{vac}\rangle = p|\mathrm{vac}\rangle \tag{2.22}$$

gives the *definition* of suitable vacuum $|\mathrm{vac}\rangle$ in the space-time CFT of GKS (which will be called as *space-time vacuum* in this paper). Of course $|\mathrm{vac}\rangle$ need not (and should not) be equal to $|0\rangle$. Although the space-time CFT is defined by the degrees of freedom on world-sheet, *it is not the AdS_3-string theory itself*, and we have no contradiction if we take a different vacuum $|\mathrm{vac}\rangle$ for this theory.

In the last section we will again give some comments to clarify the compatibility between the GKS's construction of space-time CFT and the work [16].

[3]We work in a convention in which the definition of the third of the five signs is opposite to that of ref. [1].

3. Vacuum of the spacetime CFT

3.1 Space-time vacuum with $p = 1$

It is an important problem to find out the vacuum vector of the space-time CFT. Clearly, we must find it in the physical Hilbert space of string theory, and it should have the superconformal invariance with respect to the superconformal algebra of the space-time CFT. Moreover, it should be emphasized that it is indeed different from the usual vacuum of superstring theory, as we already commented at the last of previous section. This is because the condition $\oint dz\gamma^{-1}\partial\gamma = p$ must be realized on this vacuum.

Namely, we shall impose on the vacuum $|\mathrm{vac}\rangle$ the following conditions:

1. $|\mathrm{vac}\rangle$ is BRST invariant in ths sense of superstring theory,

$$Q_{\mathrm{BRST}}|\mathrm{vac}\rangle = 0. \tag{3.1}$$

2. $|\mathrm{vac}\rangle$ is primary and has the global superconformal invariance in the sense of space-time CFT (up to some BRST-exact terms).

$$\begin{aligned} \mathcal{L}_n\,|\mathrm{vac}\rangle &= 0\ ; \qquad (n \geq -1) \\ \mathcal{G}_r^{\alpha A}\,|\mathrm{vac}\rangle &= 0\ ; \qquad \left(r \geq -\frac{1}{2}\right) \\ \mathcal{K}_n^a\,|\mathrm{vac}\rangle &= 0\ ; \qquad (n \geq 0) \end{aligned} \tag{3.2}$$

3. The vacuum has the winding number p of γ.

$$\oint dz\gamma^{-1}\partial\gamma\,|\mathrm{vac}\rangle = p\,|\mathrm{vac}\rangle \tag{3.3}$$

Let us start with a natural ansatz of the form $|\mathrm{vac}\rangle = cV(0)\,|0\rangle$, where c denotes the reparametrization ghost (spin -1) and $V(z)$ should be some primary field with conformal weight 1, which will be constructed below.

To this aim (in particular, to realize the third condition (3.3)) it is useful to introduce the following scalar fields X, Y bosonizing the $\beta\gamma$-system (of AdS_3 part of worldsheet theory); as

$$\beta = i\partial Y e^{-Z}\,, \tag{3.4}$$

$$\gamma = e^{Z}\,, \tag{3.5}$$

$$Z = X + iY\,, \qquad X(z)X(w) \sim Y(z)Y(w) \sim -\ln(z-w)\,. \tag{3.6}$$

We can easily obtain $\gamma^{-1}\partial\gamma = \partial Z$. By this relation, it seems plausible to take the vertex operators such as $\sim e^{ipY}$ or $\sim e^{-pX}$ as candidates for $V(z)$. Unfortunately, the requirements of the BRST invariance (3.1) and superconformal invariance in space-time theory (3.2) are very strong conditions, and hence it seems that almost all possibilties are excluded in the cases of generic p. The best we can do here is to set $p = 1$ for the time being. We will later discuss the cases with generic p.

Now, we can easily find that the operator e^{iY} is actually a primary field with conformal weight 1 under the correct background charge fixed by that of the $U(1)$-current $\beta\gamma$. Also we can check that this commutes with all the zero-modes of bosonic $SL(2, \mathbf{R})$ currents up to total derivative terms[4] (It is obvious that this commutes with all of the fermionic currents and the bosonic $SU(2)$-currents.) It immediately gives the superconformal invariance (3.2). The condition of BRST invariance is more non-trivial. As we already know this candidate has a suitable conformal weight, we must only confirm the invariance under the fermionic part of BRST-transformation (including the supercurrent $G(z)$ in the worldsheet). Unfortunately, we find that this is not actually BRST-invariant. But a careful analysis gives us the following completion possessing the full BRST-invariance:

$$|\mathrm{vac}\rangle = ce^{iY}(0)\,|0\rangle + \frac{1}{\sqrt{2k}}\eta\, e^{\phi}e^{-X}\psi^{+}\,|0\rangle\ . \tag{3.7}$$

In the above expression, $\eta(z)$ denotes a fermionic field with spin 1 composing the bosonized super-ghost system together with the fermionic partner ξ (spin 0) and the bosonic field ϕ already introduced in the previous section [21]. One can show that this state satisfies all of the above conditions for the *space-time vacuum* (3.1), (3.2) and (3.3), and is no other than our proposal of space-time vacuum in the case of $p = 1$.

There is one comment which may be useful for some explicit calculations: The second term in the RHS of (3.7) vanishes in almost all cases when some vertex operators with a negative picture (for example, operators in the (-1)-picture for NS sector, and in the $(-1/2)$-picture for R sector, which are *standard* pictures we often use) act on it, although we need this term to ensure the BRST-invariance.

3.2 The space-time vacuum with general p

Now, how can we construct the vacuum for the general cases with $p > 1$?

To answer this question, we would like to make the following proposal; *We should identify the Hilbert space of space-time CFT with (at least, a subspace of) the physical*

[4] There is only one non-trivial OPE;

$$j^{-}(z)e^{iY}(0) \sim \frac{1}{z^2}e^{-X}(0) + \frac{1}{z}\partial(e^{-X})(0).$$

The existence of the term with $\frac{1}{z^2}$ singularity in this OPE is crucial for our arguments with respect to the BRST invariance given below.

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Hilbert space of p-string states in second quantized superstring theory. In other words, we should consider p-copies of the world-sheet of superstring to define the space-time CFT. In this set up we can construct a suitable vacuum in the following manner

$$|\mathrm{vac}\rangle \stackrel{\mathrm{def}}{=} \frac{1}{p!}\sum_{\sigma\in S_p}\bigotimes_{i=1}^{p}|\mathrm{vac}\rangle^{(\sigma(i))} \;\in \left[(\mathcal{H}_{\mathrm{phys}})^{\otimes p}\right]^{S_p}, \tag{3.8}$$

where each $|\mathrm{vac}\rangle^{(i)}$ is defined as (3.7) for each world-sheet, and $\mathcal{H}_{\mathrm{phys}}$ stands for the physical Hilbert space of single string. The superscript S_p expresses the S_p-invariant subspace.

General operators in space-time CFT (generators of SCA, and some primary operators, and so on) should act on this Hilbert space in a natural way; $\mathcal{A} \equiv \sum_i 1\otimes 1\otimes\cdots\otimes\mathcal{A}^{(i)}\otimes\cdots\otimes 1$, where $\mathcal{A}$ denotes an arbitrary operator in space-time CFT and $\mathcal{A}^{(i)}$ acts only on the the i-th factor of $(\mathcal{H}_{\mathrm{phys}})^{\otimes p}$. It is obvious that (3.8) has all of the desired properties. Especially, we can obtain the correct winding number of γ;

$$\oint dz\gamma^{-1}\partial\gamma\,|\mathrm{vac}\rangle = \sum_{i=1}^{p} 1|\mathrm{vac}\rangle = p\,|\mathrm{vac}\rangle\,, \tag{3.9}$$

and of course the correct central charge $c = \sum_{i=1}^{p} c^{(i)} \equiv 6pk$.

One might feel this ansatz for the Hilbert space of the space-time CFT as somewhat artificial. One must, however, remember that, in the standard argument of AdS/CFT-correspondence, the Hilbert space of boundary CFT should include multi-particle states of bulk-theory. Moreover, we can show the following rough estimation of physical degrees of freedom which supports our above proposal on the Hilbert space: As the representation theory of current algebra, the world-sheet CFT has level $\sim k$, and so we should have an unitarity bound for $SU(2)$ (or $SL(2,\mathbf{R})$) charge of the order k [22]. On the other hand, the space-time CFT should have the unitarity bound for R-charge of the order pk, which is known as the name *stringy exclusion principle* [8]. In this way, one can find that the space-time CFT includes roughly p-times larger degrees of freedom than those on a *single* world-sheet. One of the most simple (and perhaps, natural) solutions to fill this gap of degrees of freedom is no other than introducing p-strings Hilbert space!

If one accept the above proposal, that is, *"space-time CFT should describe the second quantized superstring theory"*, it may be more natural to consider the Matrix string theory proposed in [23] instead of the *old-fashoned* fundamental string. From this viewpoint it may be important to consider general *twisted sectors.* (On the other hand, the simplest construction of Hilbert space given above corresponds to the *untwisted sector.*) To explain this, let us consider the following setup: Let $(n_1,\ldots,n_l)$

be an arbitrary partition of p, Namely, $n_1 \geq n_2 \geq \cdots \geq n_l > 0$ with $\sum_{i=1}^{l} n_i = p$. Then, we can introduce the single-string Hilbert space for the $\mathbf{Z}_{n_i}$*-twisted string* $\mathcal{H}_{\rm phys}^{(n_i)} \left(\cong \mathcal{H}_{\rm phys}^{\mathbf{Z}_{n_i}}\right)$ for each n_i, which is defined by the *screwing procedure*, something like

$$X^{\mu}_{(i)}\left(e^{2\pi i}z\right) = X^{\mu}_{(i+1)}(z)\,, \tag{3.10}$$

and by imposing $\mathbf{Z}_{n_i}$-invariance.

In this setup, our space-time vacuum should be constructed in the Hilbert space $\left[\bigotimes_{i=1}^{l} \mathcal{H}_{\rm phys}{}^{(n_i)}\right]^{\Gamma_{n_1,\ldots,n_l}}$, where $\Gamma_{n_1,\ldots,n_l}$ means the symmetry group composed of arbitrary permutations among twisted strings of the same *length*.

However, if one tries to construct our space-time vacuum naively for each world-sheet of $\mathbf{Z}_{n_i}$-twisted string, one might feel that our above discussion leads to a contradiction. Since we now have only $l(< p)$ worldsheets, it seems that the total central charge of the space-time CFT becomes $c = \sum_{i=1}^{l} c^{(i)} = 6lk \neq 6pk$. We should here again enphasize that in each world-sheet of l strings, our space-time vacuum $|{\rm vac}\rangle^{(i)}$ should have the property: $\oint dz \gamma^{-1}\partial\gamma\, |{\rm vac}\rangle^{(i)} = |{\rm vac}\rangle^{(i)}$. We know no solution possessing the winding number $p > 1$ *which is assigned to a single worldsheet* under the requirements of BRST-invariance and superconformal invariance of space-time. How can we overcome this difficulty?

The simplest possibilty is as follows: *we should assign the level $n_i k$ current algebras instead of k for each world-sheet of $\mathbf{Z}_{n_i}$-twisted string.* If this is indeed the case, each worldsheet yields the contribution $c^{(i)} = 6n_i k$, and we can obtain the correct central charge $c = \sum_{i=1}^{l} c^{(i)} = 6pk$. Furthermore, we will observe in the next section, how this claim about the enhancement of level gives us the suitable spectrum of chiral primary operators in the space-time CFT.

In the rest of this section, let us give a heuristic explanation about how this enhancement occurs for the world-sheets of twisted strings. For a complete understnding of it, we may have to formulate carefully the Matrix string theory on AdS_3-background. We would like to study this problem intensively in future work.

Let us consider only the maximally twisted sector to avoid unnecessary complexity. That is, assume that there is only a single world-sheet of p-joined string $\Sigma(\cong \mathbf{P}^1)$. Since our string is now the *longest string* possessing the p-times longer length, let us consider the following covering

$$\Phi : z \in \Sigma \;\longmapsto\; z^p \in \hat{\Sigma}(\cong \mathbf{P}^1)\,, \tag{3.11}$$

in order to *normalize* the unit of excitations.

Imagine some current algebra $J^a(z)$ is defined on Σ with level k. We would like to construct a proper current $\hat{J}^a(\hat{z})$ defined on the *normalized* world-sheet $\hat{\Sigma}$, on which we should formulate our space-time CFT, so that the relation $\Phi^*(\hat{J}^a d\hat{z}) = J^a\, dz$ holds. However, $J^a(z)$ cannot act suitably on $\mathcal{H}^{(p)}_{\rm phys} \cong \mathcal{H}_{\rm phys}{}^{\mathbf{Z}_p}$, because $J^a(z)\, dz$ does not possess $\mathbf{Z}_p$-invariance. Or equivalently, one can also say that J^a does not yield by this simple relation the current $\hat{J}^a$ which is *a single valued 1-form on* $\hat{\Sigma}$

Therefore we must instead adopt the follwoing definition of $\hat{J}^a$:

$$\Phi^*\left(\hat{J}^a d\hat{z}\right) = \sum_{n=0}^{p-1} \Omega^{(n)*}(J^a dz)\,, \tag{3.12}$$

where $\Omega^{(n)} : z \longmapsto e^{2\pi n i/p} z$ is a $\mathbf{Z}_p$-rotation on Σ. It is easy to see that this current $\hat{J}^a$ has actually the level kp, which is our claim to be proved.

We also comment on a simple evaluation of the degrees of freedom. Consider first the untwisted sector, namely, we have p worldsheets of *short* strings, and each worlsheet has a level k current algebra. If these p strings are completely joined one after another by interactions of Matrix string theory, we will get only one long string. The simplest and maybe the most plausible way to impose the invariance of degrees of freedom through this joining process is to require the level of current algebras on the world-sheet of long string should be enhanced to the value pk.

4. Comparison with the SUSY σ-Model on T^{4kp}/S^{kp}

In the well-known arguments of AdS_3/CFT_2-duality, it is belived that the SCFT defined as the $N = (4,4)$ supersymmetric sigma model on a symmetric orbifold T^{4kp}/S^{kp} is one of the most powerful candidates of the boundary CFT. In this sense it is an important task to compare the space-time CFT with this T^{4kp}/S^{kp} SCFT. We already know that both of them have $N = (4,4)$ superconformal symmetry with the same central charge $c = 6pk$. But, in order to establish the equivalence (or some relation) between them, we must still clarify the spectrum of these SCFTs, especially, the spectrum of chiral primaries.

In this section, we first explore further the operator algebra in the space-time CFT. We would like to find the complete set of the bosonic and fermionic oscillators inspired by the T^{4kp}/S^{kp}-theory (the Heisenberg algebras along T^4).

Next, we will discuss the issue of the physical spectrum of chiral primaries in the space-time CFT. We will observe how we can obtain the spectrum which is equivalent with that of T^{4kp}/S^{kp}-SCFT in the Hilbert space of space-time CFT proposed in the previous section.

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4.1 Construction of mode operators along T^4

In the σ-model on T^{4kp}/S^{kp} there is a *diagonal* T^4 which is free from any operations of orbifolding. Therefore the spacetime CFT contains a sector which is nothing but a free SCFT on T^4. And we can find rather easily the corresponding degree of freedom in the worldsheet theory.

Define $\left\{a_n^{AK}, x_n^{AK}, b_r^{\alpha K}\right\}$ as

$$\begin{aligned}
ia_n^{++} &= -\sqrt{\frac{k}{2}}\oint dze^{-\phi}\psi^{+5}\gamma^n \sim -\sqrt{\frac{k}{2}}\oint dzi\partial\left(Y^3+iY^4\right)\gamma^n,\\
ia_n^{+-} &= +\sqrt{\frac{k}{2}}\oint dze^{-\phi}\psi^{+4}\gamma^n \sim +\sqrt{\frac{k}{2}}\oint dzi\partial\left(Y^1+iY^2\right)\gamma^n,\\
ia_n^{-+} &= +\sqrt{\frac{k}{2}}\oint dze^{-\phi}\psi^{-4}\gamma^n \sim +\sqrt{\frac{k}{2}}\oint dzi\partial\left(Y^1-iY^2\right)\gamma^n,\\
ia_n^{--} &= +\sqrt{\frac{k}{2}}\oint dze^{-\phi}\psi^{-5}\gamma^n \sim +\sqrt{\frac{k}{2}}\oint dzi\partial\left(Y^3-iY^4\right)\gamma^n,
\end{aligned}\tag{4.1}$$

where readers must understand "$\sim$" in the sense of equivalence by the picture changing. Similarly, we define

$$\begin{aligned}
x_n^{++} &= +\frac{1}{2}\oint dze^{-\phi}\left(Y^3+iY^4\right)\gamma^n\left(\gamma\psi^- + \gamma^{-1}\psi^+ - 2\psi^3\right),\\
x_n^{+-} &= -\frac{1}{2}\oint dze^{-\phi}\left(Y^1+iY^2\right)\gamma^n\left(\gamma\psi^- + \gamma^{-1}\psi^+ - 2\psi^3\right),\\
x_n^{-+} &= -\frac{1}{2}\oint dze^{-\phi}\left(Y^1-iY^2\right)\gamma^n\left(\gamma\psi^- + \gamma^{-1}\psi^+ - 2\psi^3\right),\\
x_n^{--} &= -\frac{1}{2}\oint dze^{-\phi}\left(Y^3-iY^4\right)\gamma^n\left(\gamma\psi^- + \gamma^{-1}\psi^+ - 2\psi^3\right),
\end{aligned}\tag{4.2}$$

$$\begin{aligned}
b_r^{++} &= +i\oint dze^{-\frac{\phi}{2}}\left[\gamma^{r-\frac{1}{2}}S^{+++-+} + \gamma^{r+\frac{1}{2}}S^{-+--+}\right],\\
b_r^{+-} &= +i\oint dze^{-\frac{\phi}{2}}\left[\gamma^{r-\frac{1}{2}}S^{++++-} + \gamma^{r+\frac{1}{2}}S^{-+-+-}\right],\\
b_r^{-+} &= -i\oint dze^{-\frac{\phi}{2}}\left[\gamma^{r-\frac{1}{2}}S^{+---+} + \gamma^{r+\frac{1}{2}}S^{--+-+}\right],\\
b_r^{--} &= -i\oint dze^{-\frac{\phi}{2}}\left[\gamma^{r-\frac{1}{2}}S^{+--+-} + \gamma^{r+\frac{1}{2}}S^{--++-}\right].
\end{aligned}\tag{4.3}$$

The expressions (4.1) are already given in [1], and (4.2), (4.3) are our original results.

One can verify the following commutation relations

$$\begin{aligned}
\left[a_m^{AK}\ ,\ a_n^{BL}\right] &= -kpm\delta_{m+n}\epsilon^{AB}\epsilon^{KL}\,,\\
\left[x_m^{AK}\ ,\ a_n^{BL}\right] &= kp\delta_{m+n}\epsilon^{AB}\epsilon^{KL}\,,\\
\left[x_m^{AK}\ ,\ x_n^{BL}\right] &= 0\,,\\
\left\{b_r^{\alpha K}\ ,\ b_s^{\beta L}\right\},\ &= kp\delta_{r+s}\epsilon^{\alpha\beta}\epsilon^{KL}\,,
\end{aligned}\tag{4.4}$$

where $\epsilon^{+-} = -\epsilon^{-+} = 1$ for any kind of indices. They also *correctly* act on our space-time vacuum:

$$a_n^{AK}|\mathrm{vac}\rangle = 0\,, \qquad (\forall n \geq 0)\,.$$
$$b_r^{\alpha K}|\mathrm{vac}\rangle = 0\,, \qquad (\forall r \geq 1/2)\,. \tag{4.5}$$

Furthermore we can prove by straightforward (but, the parts including the OPEs with respect to spin fields are rather complicated) calculations that they have the following commutation relations with the SCA generators:

$$\begin{aligned}
\left[\mathcal{L}_m\,,\, a_n^{AL}\right] &= -n a_{m+n}^{AL}\,,\\
\left[\mathcal{K}_m^a\,,\, a_n^{AL}\right] &= 0\,,\\
\left[\mathcal{G}_r^{\alpha A}\,,\, a_n^{BK}\right] &= \epsilon^{AB} b_{n+r}^{\alpha K}\,,\\
\left[\mathcal{L}_m\,,\, x_n^{AL}\right] &= -n x_{m+n}^{AL}\,,\\
\left[\mathcal{K}_m^a\,,\, x_n^{AL}\right] &= 0\,,\\
\left[\mathcal{G}_r^{\alpha A}\,,\, x_n^{BK}\right] &= 0\,,\\
\left[\mathcal{L}_m\,,\, b_s^{\alpha K}\right] &= -\left(\frac{m}{2}+s\right) b_{m+s}^{\alpha K}\,,\\
\left[\mathcal{K}_m^a\,,\, b_s^{\alpha K}\right] &= \frac{1}{2}\left(\sigma^a\right)^{\alpha}_{\ \beta} b_{m+s}^{\beta K}\,, \qquad \sigma^a = \begin{pmatrix} (\sigma^a)^-_{\ -} & (\sigma^a)^-_{\ +}\,, \\ (\sigma^a)^+_{\ -} & (\sigma^a)^+_{\ +} \end{pmatrix},\\
\left\{\mathcal{G}_r^{\alpha A}\,,\, b_s^{\beta K}\right\} &= \epsilon^{\alpha\beta} a_{r+s}^{AK}\,.
\end{aligned} \tag{4.6}$$

In this way we have found that we can safely identify $\left\{a_n^{AK}, x_0^{AK}, b_r^{\alpha K}\right\}$ with the mode operators of SCFT on diagonal T^4. As for x_n^{AK} with $n \neq 0$, we have still no idea on their role in the spacetime CFT. We hope to discuss elsewhere more about them.

One can also construct more general oscillators in a trivial manner, at least on the Hilbert space of untwisted sector. Let $\mathcal{A}$ be any of $a_n^{AK}, x_0^{AK}, b_r^{\alpha K}$. We can take $\mathcal{A}^{(i)}$ $(i = 1, \ldots, p)$ (which was already defined as the restriction of $\mathcal{A}$ on the i-th factor of our untwisted sector Hilbert space) as the desired mode operator. It is obvious that such oscillators satisfy the same (anti-)commutation relations as those of T^{4p}/S_p SCFT with $c = 6p$.

Let us consider the case $k = 1$. In the space-time CFT, we have got the bosonic and fermionic mode operators which have the same algebraic structure as those of fundamental fields of T^{4p}/S_p SCFT. From these oscillators, we can further construct a $N = 4$ SCA in the well-known quadratic forms of oscillators. Now, a natural question arises: Is this SCA the same one as the SCA defined in section 2?

Let $\mathcal{G}_{\mathrm{GKS}}$ be any SCA generator introduced in section 2, and $\mathcal{G}_{\mathrm{quad}}$ be the corresponding operator defined in the quadratic form by the oscillators. Our question is whether or not $\mathcal{G}_{\rm GKS}\,|\alpha\rangle = \mathcal{G}_{\rm quad}\,|\alpha\rangle$ holds for any state $|\alpha\rangle$ in our space-time Hilbert space.

We already know that any $\mathcal{G}_{\rm GKS}$ and $\mathcal{G}_{\rm quad}$ have the same commutaiton relations with any of our oscillators. So, the operator $\mathcal{G}_{\rm GKS} - \mathcal{G}_{\rm quad}$ commutes with all of the oscillators. It immediately follows from this fact that $\mathcal{G}_{\rm GKS}\,|\alpha\rangle = \mathcal{G}_{\rm quad}\,|\alpha\rangle$ holds *up to null states* as far as our Hilbert space can be spanned by these oscillators. In a few simple examples, we actually face the situations that the differences between them do *not* vanish. But even in these cases we can observe by direct calculation that they become truly spurious states up to some BRST exact terms.

4.2 Chiral primaries

Now we study the spectrum of chiral primaries of the space-time CFT and compare it with that of the T^{4kp}/S^{kp}-SCFT. We here set $p=1$ for the time being.

It is a famous fact about the symmetric orbifold of this type that there is a sequence of chiral primaries

$$\omega^{q,\bar{q}}_{(j)}(z,\bar{z}) = \omega^{q}_{(j)}(z)\bar{\omega}^{\bar{q}}_{(j)}(\bar{z})\,, \qquad j = 0,\, \frac{1}{2},\, 1,\, \ldots$$

for each element $\omega^{q,\bar{q}}$ of cohomology of T^4, and this operator has the R-charge $(Q,\bar{Q}) = (\frac{q}{2}+j, \frac{\bar{q}}{2}+j)$. The above kind of chiral primaries will be called as those of *single-particle type*. In addition to these one can obtain many other chiral primaries of *multi-particle type* by taking products of single-particle ones. It is interesting to consider if we can find the same spectrum of chiral primaries in the space-time CFT.

Let us begin with the chiral primaries with $j=0$. One can find out the following correspondence:

$$\begin{aligned}
(q=0) &\longleftrightarrow \left(\omega^{0}_{(0)}\right)_n = \delta_{n,0}\,,\\
(q=1) &\longleftrightarrow \left(\omega^{1,\pm}_{(0)}\right)_r = b^{+\pm}_r\,,\\
(q=2) &\longleftrightarrow \left(\omega^{2}_{(0)}\right)_n = \mathcal{K}^{+}_n\,,
\end{aligned} \tag{4.7}$$

or equivelently, we can also express them as the chiral primary *states*;

$$\begin{aligned}
\left|\omega^{0}_{(0)}\right\rangle &= \gamma^{-1}\psi^{+}e^{-\phi}(0)|{\rm vac}\rangle\,,\\
\left|\omega^{1,\pm}_{(0)}\right\rangle &= b^{+\pm}_{-1/2}|{\rm vac}\rangle \equiv i\gamma^{-1}S^{+++\mp\pm}e^{-\frac{\phi}{2}}(0)|{\rm vac}\rangle\,,\\
\left|\omega^{2}_{(0)}\right\rangle &= \mathcal{K}^{+}_{-1}|{\rm vac}\rangle \equiv \gamma^{-1}\chi^{+}e^{-\phi}(0)|{\rm vac}\rangle\,,
\end{aligned} \tag{4.8}$$

where readers should understand the product such as "$\gamma^{*}|{\rm vac}\rangle$" in the sense of the normal product. (Remember $|{\rm vac}\rangle$ includes e^{iY} in the definition (3.7).) Of course, these states are BRST-invariant and satisfy the following relations characterizing the chiral primary states;

$$Q_{\mathrm{BRST}}\left|\omega^q_{(0)}\right\rangle = 0 \tag{4.9}$$

$$\begin{aligned}
\mathcal{L}_n\left|\omega^q_{(0)}\right\rangle &= 0 \qquad (\forall n \geq 1)\,,\\
\mathcal{K}^a_n\left|\omega^q_{(0)}\right\rangle &= 0 \qquad (\forall n \geq 1)\,,\\
\mathcal{G}^{+A}_r\left|\omega^q_{(0)}\right\rangle &= 0 \qquad \left(\forall n \geq -\frac{1}{2}\right),\\
\mathcal{G}^{-A}_r\left|\omega^q_{(0)}\right\rangle &= 0 \qquad \left(\forall n \geq \frac{1}{2}\right).
\end{aligned} \tag{4.10}$$

We also obtain

$$\mathcal{L}_0\left|\omega^q_{(0)}\right\rangle = \mathcal{K}^3_0\left|\omega^q\right\rangle = \frac{q}{2}\left|\omega^q_{(0)}\right\rangle. \tag{4.11}$$

We shall call it the "chiral primary state with R-charge $\frac{q}{2}$".

It is clear that the chiral primaries with $j = 0$ (4.7), (4.8) should be identified with those within the untwisted sector of T^{4kp}/S^{kp}-SCFT.

To write down the chiral primaries of single-particle type for generic j one needs some preparations. Let us define the following operators on the worldsheet

$$V_{j,m} \equiv \gamma^{j+m}\exp\left[\frac{2j\varphi}{\alpha_+}\right], \tag{4.12}$$

$$\tilde{V}_{j,m} \equiv \frac{1}{\sqrt{(j+m)!(j-m)!}}\tilde{\gamma}^{j+m}\exp\left[\frac{-2ij\tilde{\varphi}}{\alpha_+}\right], \tag{4.13}$$

which have the following OPE's with $SL(2,R)$ and $SU(2)$ currents

$$\begin{aligned}
j^3(z)V_{j,m}(w) &\sim \frac{m}{z-w}V_{j,m}(w)\,,\\
j^{\pm}(z)V_{j,m}(w) &\sim \frac{m\mp j}{z-w}V_{j,m\pm 1}(w)\,,\\
k^3(z)\tilde{V}_{j,m}(w) &\sim \frac{m}{z-w}\tilde{V}_{j,m}(w)\,,\\
k^{\pm}(z)\tilde{V}_{j,m}(w) &\sim \frac{1}{z-w}\sqrt{(j\mp m)(j\pm m+1)}\tilde{V}_{j,m\pm 1}(w)\,.
\end{aligned} \tag{4.14}$$

Then we can find a chiral primary for each q, having the following operators as modes:

$$\begin{aligned}
q=0 \longleftrightarrow \left(\omega^0_{(j)}\right)_n &= \oint dz e^{-\phi}\tilde{V}_{j,j}V_{j,n}\left(\gamma\psi^- + \gamma^{-1}\psi^+ - 2\psi^3\right),\\
q=1 \longleftrightarrow \left(\omega^{1,\pm}_{(j)}\right)_r &= \oint dz e^{-\frac{\phi}{2}}\tilde{V}_{j,j}\left[V_{j,r-\frac{1}{2}}S^{+++\mp\pm} + V_{j,r+\frac{1}{2}}S^{-+-\mp\pm}\right],\\
q=2 \longleftrightarrow \left(\omega^2_{(j)}\right)_n &= \oint dz e^{-\phi}\tilde{V}_{j,j}V_{j,n}\chi^+\,.
\end{aligned} \tag{4.15}$$

We can also express them as the chiral primary states;

$$\begin{aligned}
\left|\omega^0_{(j)}\right\rangle &= e^{-\phi}\psi^+\tilde{V}_{j,j}V_{j,-j-1}(0)|\mathrm{vac}\rangle\,, \\
\left|\omega^{1,\pm}_{(j)}\right\rangle &= ie^{-\frac{\phi}{2}}S^{+++\mp\pm}\tilde{V}_{j,j}V_{j,-j-1}(0)|\mathrm{vac}\rangle\,, \\
\left|\omega^2_{(j)}\right\rangle &= e^{-\phi}\chi^+\tilde{V}_{j,j}V_{j,-j-1}(0)|\mathrm{vac}\rangle\,.
\end{aligned} \tag{4.16}$$

There are some comments here: First, in tensoring the chiral primaries of left and right movers one has to set the quantum number j for the both movers equal. This is because one can think of j as the momentum of φ, which parameterizes the radial direction of AdS_3 and is evidently non-compact. This aspect about the R-charge is consistent with the known results about the spectrum of cohomology of symmetric orbifolds.

Second, let us discuss about the unitarity bound for the R-charge of chiral primaries. In the above constuction of chiral primaries, the value of the quantum number j should have an upper-bound $(k-2)/2 \sim k/2$ [22, 1]. On the other hand, it is known that the unitarity bound for the chiral primaries of the *single* pariticle type in the T^{4kp}/S^{kp}-SCFT is equal to $\sim kp/2$.[5] As was already claimed by several authors [22, 1, 15], in the case of $p = 1$ both of these results are consisitent. But, how about the case $p > 1$? In [1], it was discussed that the missing states with $k/2 \leq j \leq pk/2$ might be multi-particle states. However, this interpretation seems to contradict with the Kaluza-Klein spectrum in SUGRA discussed in [24]. We do need the chiral primaries of *single*-particle type possessing large R-charges beyond $k/2$.

In the previous section we proposed the Hilbert space of space-time CFT for general p. Especially, we discussed a phenomenon that the levels of $SU(2)$ and $SL(2,\mathbf{R})$ current algebras effectively enhance to pk on the worldsheet of the maximally joined string. So, we can get $pk/2$ instead of $k/2$ as the unitarity bound for the chiral primaries of single particle type, which is no other than the result we want! We can find out the single-particle chiral primary saturating the bound $pk/2$ on this worldsheet without any problem.

Let us then turn to the multi-particle type. They should be defined as some products of chiral primaries of single-particle type. We would like to discuss here the chiral primaries of multi-particle type which are obtained by multiplying those of single particle type with $j = 0$. The following discussion can be easily generalized to the cases of more complicated multi-particle types.

[5] The unitarity bound for general (namely, including multi-particle types) chiral primary is equal to $\frac{c}{6} = kp$. (Remark that the unit of R-charge we used here is 1/2 of the usual one in $N = 2$ SCFT.) This is usually called *stringy exclusion principle* [8]. But, it should be emphasized that the unitarity bound for the *single* particle type is 1/2 of it.

By taking the product of $\omega^p_{(0)}(z)$ one can obtain the following three types of multi-particle chiral primaries:

$$\left[\omega^2_{(0)}(z)\right]^i \,, \qquad \left[\omega^2_{(0)}(z)\right]^i \omega^{1,\pm}_{(0)}(z) \,, \qquad \left[\omega^2_{(0)}(z)\right]^i \omega^{1,+}_{(0)}(z)\omega^{1,-}_{(0)}(z) \,. \tag{4.17}$$

More concretely, we can also express them in terms of mode operators. For instance, the modes of the first example can be written as

$$\left[\omega^2_{(0)}(z)\right]^i_n = \sum_{n_1,\ldots,n_i} \mathcal{K}^+_{n_1}\cdots\mathcal{K}^+_{n_i}\cdot\delta_{n,n_1+\cdots+n_i} \,, \tag{4.18}$$

and the corresponding chiral primary states is as follows;

$$\left|\left[\omega^2_{(0)}(z)\right]^i\right\rangle = \left(\mathcal{K}^+_{-1}\right)^i |\mathrm{vac}\rangle. \tag{4.19}$$

Here we give also a comment about the spectrum. Let us first consider the case of $p=1$. In this expression (4.19), we have an upper-bound k for the value i, because $\left(\mathcal{K}^+_{-1}\right)^{k+1} |\mathrm{vac}\rangle = 0$ (null state) holds.

On the other hand, the corresponding operators in the T^{4kp}/S_{kp} SCFT can be expressed in the simple form of products of fermionic oscillators. (Recall that $\omega^2_{(0)}$ corresponds to the operator of the form $\sim \sum_{A=1}^{kp} \Psi^{++}_A \Psi^{+-}_A$.) Hence we can obtain the bound $i \leq kp$ which is due to nothing but the fermi-statistics. We again find both are consisitent, if $p=1$.

To treat the cases with $p>1$, let us consider the untwisted sector of space-time Hilbert space, which is nothing but a symmetrized tensor product of single-string Hilbert space. We can immediately discover the desired multi-particle chiral primary saturating the unitarity bound kp in this Hilbert space. In fact, we only have to consider p-symmetrized tensor product of $\left(\omega^2_{(0)}\right)^k$. (In other words, one can simply imagine the situation that k-particle chiral primary state lives on each worldsheet of p short strings, which gives a kp-particle chiral primary state.) We can also present the similar arguments for the second and third examples of multi-particle chiral primaries (4.17).

In this way, we have obtained a satisfactory correspondence between the chiral primaries in the space-time CFT and those of T^{4kp}/S_{kp} SCFT.

5. Conclusions and discussions

In this paper, according to the framework presented in [1], we investigated the space-time CFT associated with the superstring on the background $AdS_3 \times S^3 \times T^4$, which is an interesting proposal of boundary CFT in the AdS_3/CFT_2-correspondence. We explicitly constructed the suitable vacuum of the Hilbert space of the space-time CFT in the case $p = 1$. In the general cases with $p > 1$, we proposed that

$$\mathcal{H}_{\text{space-time}} = \bigoplus_{\mathbf{n}\in Y_p} \left[\bigotimes_{i=1}^{l(\mathbf{n})} \mathcal{H}_k^{\mathbf{Z}_{n_i}}\right]^{\Gamma_{\mathbf{n}}} . \tag{5.1}$$

In this expression, $\mathbf{n} \equiv (n_1, \ldots, n_l) \in Y_p$ means any partition of p (namely, Y_p denotes the set of Young tableaus composed of p boxes), $l \equiv l(\mathbf{n})$ expresses the *depth* of tableau $\mathbf{n}$. $\Gamma_{\mathbf{n}}$ is the symmetry group composed of any permutations of twisted strings with the same length as before. $\mathcal{H}_k$ expresses the Hilbert space associated to a single AdS_3-string with level k for the WZW sectors, in which we first constructed the space-time vacuum in section 3. We also discussed about an effective enhancement of the level on the worldsheet of twisted strings, and we observed in section 4, how this enhancement of level can resolve the problems about the chiral primaries. In order to confirm our proposal about space-time Hilbert space, it may be significant to formulate carefully the Matrix string theory on AdS_3-background, which will become our most important task in future.

At this stage we would like again to make some comments on the compatibilities of the work of GKS and the work [16]. In the framework given in [1, 16] the fundamental claim of AdS_3/CFT_2-duality may be encoded somewhat formally as follows;

$$\begin{aligned}
&\langle \mathcal{O}_1(\varphi,\beta,\gamma;x_1)\mathcal{O}_2(\varphi,\beta,\gamma;x_2)\cdots\mathcal{O}_n(\varphi,\beta,\gamma;x_n)\rangle_{\text{String on } AdS_3} \\
&\qquad = \langle \text{vac}|\text{T}\left[\mathcal{O}_1(\varphi,\beta,\gamma;x_1)\mathcal{O}_2(\varphi,\beta,\gamma;x_2)\cdots\mathcal{O}_n(\varphi,\beta,\gamma;x_n)\right]|\text{vac}\rangle\,,
\end{aligned} \tag{5.2}$$

where $\mathcal{O}_i(\varphi,\beta,\gamma;x)$ stands for some chiral primary operator of the boundary theory inserted at the point $x \in$ boundary.[6] One must remember that $\mathcal{O}_i(\varphi,\beta,\gamma;x)$ is made up of the degrees of freedom of string theory in bulk. So, one can calculate these correlation functions in two ways, *as string theory in bulk and as the space-time CFT*. Namely, the left hand side of the above expression (5.2) should be evaluated in the string theory on AdS_3 back-ground. This includes the integral over the moduli of worldsheets (quantization of two-dimensional gravity) and we must sum over all the different topology of worldsheets, including the non-connected ones.

On the other hand, in the right hand side the correlator among the *same* operators $\mathcal{O}_i$ must be computated in terms of the operator algebra in the space-time CFT, which is derived from the fundamental OPE's of $\varphi, \beta, \gamma, \ldots$, and our definition of space-time vacuum $|\text{vac}\rangle$. "T" denotes the T-order (radial order) defined with

[6]Although in [1] and this paper only the expressions of mode oscillators of chiral primaries are given, we can always define this operator $O_i(x)$ at least in a formal sense.

respect to the boundary points $x_1, \ldots, x_n$. Here we again emphasize that the space-time CFT is *not* a string theory *but* a two-dimensional CFT defined on the *fixed* worldsheet (that is, the boundary of AdS-space), although it is made up of the fundamental field contents of the string theory in bulk. For the calculation in the right hand side we need not integrate moduli of worldsheet and not take the summation with respect to the topology of worldsheets.

One can say that the analyses in [16] are concerned with the left hand side of (5.2) and our work and that of [1] are studies about the right hand side. Both of them may be compatible in this sense. Of course, in order to justify this compatibility completely, one must prove the identity (5.2) for arbitrary chiral primaries, at least in the semi-classical level of bulk theory. It will be a very important and challenging problem in our future works. In any case we believe that studies along the lines of [1] and [16] will play some complementary roles to each other in understanding of the AdS_3/CFT_2-duality.

There are also some comments about the other open problems. First, it is an important task to analyse more about the physical spectrum (BRST cohomology) of AdS_3-string theory, at least, for the chiral primaries (in the sense of space-time, of course). We here only point out the following: Because the Virasoro generators of the world-sheet and space-time CFTs are different, the meanings of *primary field* for both theories are different in general. So, our analysis in section 4 is not complete, since we merely analyzed the states which are primary not only in the sense of space-time but also in the sense of world-sheet.

Second, we must remark the fact that the space-time conformal algebra proposed by GKS does *not* commute with the screening charges in the $SL(2,\mathbf{R})$-sector (especially, for the higher modes). Therefore, precisely speaking, we may have to regard AdS_3-string as the Wakimoto free field system itself, instead of $SL(2,\mathbf{R})$-WZW model. (Namely, our claim here is that we need not take the Bernard-Felder cohomology [25] to define the physical Hilbert space.) If so, one might have to reconsider about the unitarity of the string theory on AdS_3, since the no-ghost theorem given in [22] was proved under the assumption that the Hilbert space of $SL(2,\mathbf{R})$-sector is an irreducible module of current algebra, not the Fock module of Wakimoto free fields.

As the third comment, we would like to mention on the similarity of our space-time Hilbert space with that of T^{4kp}/S_{kp}. This can be written as

$$\mathcal{H}_{T^{4kp}/S_{kp}} = \bigoplus_{\mathbf{n}\in Y_{kp}} \left[\bigotimes_{i=1}^{l(\mathbf{n})} \mathcal{H}_{T^4}^{\mathbf{Z}_{n_i}} \right]^{\Gamma_{\mathbf{n}}}, \tag{5.3}$$

where $\mathcal{H}_{T^4}$ denotes the Hilbert space of $N=4$ SUSY σ-model on T^4 with $c=6$.

In fact, in the case $k = 1$, this has the same structure as (5.1). However, in the general cases with $k > 1$, it seems that we have a discrepancy in these structures. Very recently, in [15] it is also claimed that the spectra of massive modes with KK momenta and windings along T^4 in the space-time CFT does not coincide with those of T^{4kp}/S_{kp}-SCFT on every points of moduli space *except the case* $k = 1$. The observation here might have some relation to this claim. We wish to further discuss this problem elsewhere.

Our work in this paper was only concerned with the formulation. We would also like to apply our formalism to more *physical* problems, namely, calculations of correlation functions and their applications to BTZ black-hole physics [26]. Especially, it may be interesting to generalize our results to superstring on a BTZ black-hole background. To this aim we will have to formulate the space-time CFT on torus. If we succeed in it completely, we will be able to define the quantum theory of BTZ black-holes with finite temperature *from the stringy viewpoint*, and it will become a significant subject to compare it with the previous works on BTZ black-holes based on 3-dimensional gravity [27].

Note added. After completing the essential part of this work, we became aware of the paper [15], which has some overlap with our results in section 4.

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A. Fermions, spin fields and their OPE's

Here we shall give a definition of spin fields by their OPE's. Firstly we define a set of spinors $\psi^{\mu}(\mu = \pm 1, \pm 2, \pm 3, \pm 4, \pm 5)$ by the following equations:

$$\begin{aligned}
\psi^{\pm 1} &\equiv \psi^1 \pm i\psi^2\,, \\
\psi^{\pm 2} &\equiv \chi^1 \pm i\chi^2\,, \\
\psi^{\pm 3} &\equiv \chi^3 \pm \psi^3\,, \\
\psi^{\pm 4} &\equiv \lambda^1 \pm i\lambda^2\,, \\
\psi^{\pm 5} &\equiv \lambda^3 \pm i\lambda^4\,.
\end{aligned} \tag{A.1}$$

They have the following OPE:

$$\psi^\mu(z)\psi^\nu(w) \sim \frac{2\eta^{\mu\nu}}{z-w}\,, \tag{A.2}$$

$$\eta^{\mu\nu} \equiv \delta^{\mu+\nu}\,. \tag{A.3}$$

Secondly we introduce the spin fields $S^A(z)$ as the fields satisfying the following OPE's

$$\frac{1}{2} : \psi^\mu\psi^\nu : (z)S^A(w) \sim -\frac{1}{z-w}(\Gamma^{\mu\nu})^A{}_B S^B(w)\,, \tag{A.4}$$

where $\Gamma^{\mu\nu} \equiv \Gamma^{[\mu}\Gamma^{\nu]}$ and Γ^μ is the ten-dimensional gamma-matrix satisfying

$$\{\Gamma^\mu, \Gamma^\nu\} = \eta^{\mu\nu}\,. \tag{A.5}$$

Ten dimensional gamma-matrix is of course 32-dimensional, and we use as spinorial indices $(A, B, \ldots)$ the sets of five signs as in the main text, namely, S^A consists of 32 components

$$S^{+++++},\ S^{++++-}, \ldots,\ S^{-----}\,.$$

As a matter of fact, half of the above 32 components play no role. Throughout this paper we restrict ourselves to the components of spin fields with the fixed chirality, namely, $S^{\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5}$ with $\Pi\epsilon_i = -1$. The OPE's of spin fields themselves are

$$S^A(z)S^B(w) \sim (z-w)^{-\frac{3}{4}}(\Gamma_\mu C)^{AB}\sqrt{\frac{k}{2}}\psi^\mu\,, \tag{A.6}$$

where C is the charge conjugation matrix.

Equations (A.4) and (A.6) are enough for us to prove all the equations in this paper, if the gamma matrices and the charge conjugation matrix have the following components:

$$\begin{aligned}
\Gamma_{\pm 1} = \Gamma^{\mp 1} &= -\sigma_\pm \otimes \sigma_3 \otimes 1 \otimes \sigma_3 \otimes 1\,, \\
\Gamma_{\pm 2} = \Gamma^{\mp 2} &= -1 \otimes \sigma_\pm \otimes \sigma_3 \otimes \sigma_3 \otimes 1\,, \\
\Gamma_{\pm 3} = \Gamma^{\mp 3} &= \sigma_3 \otimes 1 \otimes \sigma_\pm \otimes \sigma_3 \otimes 1\,, \\
\Gamma_{\pm 4} = \Gamma^{\mp 4} &= 1 \otimes 1 \otimes 1 \otimes \sigma_\pm \otimes 1\,, \\
\Gamma_{\pm 5} = \Gamma^{\mp 5} &= \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_\pm\,, \\
C &= \epsilon \otimes \epsilon \otimes \epsilon \otimes \epsilon \otimes \epsilon\,,
\end{aligned} \tag{A.7}$$

with

$$\sigma^a = \begin{pmatrix} (\sigma^a)^+{}_+ & (\sigma^a)^+{}_- \\ (\sigma^a)^-{}_+ & (\sigma^a)^-{}_- \end{pmatrix},$$

$$
\begin{aligned}
\sigma^{+} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\
\sigma^{-} &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\
\sigma^{3} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
\epsilon &= \begin{pmatrix} \epsilon^{++} & \epsilon^{+-} \\ \epsilon^{-+} & \epsilon^{--} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\end{aligned} \tag{A.8}
$$

or equivalently,

$$
\begin{aligned}
(\Gamma_{\pm 1})^{\pm\epsilon_2\epsilon_3\epsilon_4\epsilon_5}_{\mp\epsilon_2\epsilon_3\epsilon_4\epsilon_5} &= -\epsilon_2\epsilon_4\,, \\
(\Gamma_{\pm 2})^{\epsilon_1\pm\epsilon_3\epsilon_4\epsilon_5}_{\epsilon_1\mp\epsilon_3\epsilon_4\epsilon_5} &= -\epsilon_3\epsilon_4\,, \\
(\Gamma_{\pm 3})^{\epsilon_1\epsilon_2\pm\epsilon_4\epsilon_5}_{\epsilon_1\epsilon_2\mp\epsilon_4\epsilon_5} &= \epsilon_1\epsilon_4\,, \\
(\Gamma_{\pm 4})^{\epsilon_1\epsilon_2\epsilon_3\pm\epsilon_5}_{\epsilon_1\epsilon_2\epsilon_3\mp\epsilon_5} &= 1\,, \\
(\Gamma_{\pm 5})^{\epsilon_1\epsilon_2\epsilon_3\epsilon_4\pm}_{\epsilon_1\epsilon_2\epsilon_3\epsilon_4\mp} &= \epsilon_1\epsilon_2\epsilon_3\epsilon_4\,, \\
C^{\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5,\tau_1\tau_2\tau_3\tau_4\tau_5} &= \epsilon^{\epsilon_1\tau_1}\epsilon^{\epsilon_2\tau_2}\epsilon^{\epsilon_3\tau_3}\epsilon^{\epsilon_4\tau_4}\epsilon^{\epsilon_5\tau_5}\,.
\end{aligned} \tag{A.9}
$$

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